A Finite-Element Variable Time-Stepping Algorithm for the Solution of the Electromagnetic Diffusion Equation

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Abstract — The paper presents a methodology of the Backward Differentiation Formula (BDF) for the variable time-stepping finite-element discretization. The back-history damping matrix in the BDF has been solved. The developed BDF formulation was implemented for solving the Electromagnetic Diffusion equation for first order finite elements. The final discrete system has been formulated to include magnetic nonlinearity. An initial-guess prediction algorithm was adapted for fast convergence. The algorithm was programmed in C language.

I. INTRODUCTION

In the electromagnetic machinery design and analysis, the space and time electromagnetic fields at a certain instant of time are usually required. Therefore a transient analysis is necessary. Due to the complexity that represents the solution of the partial differential equation (PDE) as a function of space and time, numerical methods to find a solution are commonly used. In the last decades, the finite element method (FEM) has proved to be one of the most popular and effective numerical PDE solution technique due to its high accuracy [1]-[2]. The Backward Differentiation Formula (BDF) is a variable time-stepping method with high accuracy results [3]-[4].

In this paper, the BDF [3] is adapted to the FEM. The backward-history damping matrix is solved and the last 1-3 time-step iteration results are used for the damping and convergence. The methodology is implemented in the transient solution of the diffusion equation using 2D first-order triangular elements. Non linear parameters are taken into account in the formulation and the system is solved using the Newton-Raphson method. An initial prediction guess algorithm is included for fast convergence. The developed algorithm was programmed in C language.

II. VARIABLE TIME-STEP ALGORITHM

The electromagnetic diffusion equation is given by (1).

$$\frac{1}{\mu}\nabla^2 A = -J + \sigma\frac{\partial A}{\partial t} \qquad (1)$$

where A stands for magnetic vector potential and is z-directed in 2D, J is the current density, μ is the magnetic permeability and σ represents the electric conductivity.

Equation (1) is expressed in the space and time domain. In the FEM space discretization of (1), a linear variation of A within the finite element is used (2).

$$A = \sum_{j=1}^{3}\alpha_j A_j \qquad (2)$$

where α_j refers to the shape functions and A_j refers to the nodal potentials. The discrete time derivation of (1) can be obtained by expanding it in a Taylor series and applying the theta algorithm [2], [5]. However, an alternative and more general time discretization with the BDF algorithm with variable time-stepping approach can be used.

The residual (3) and Jacobian (4) show the space and time discretization of equation (1) for a variable time stepping.

$$R_i = -\frac{1}{4S\mu}\left[M_{ij}\right]\left\{A^{n+1}\right\}_j + \frac{J^{n+1}S}{3} + \ldots$$

$$\frac{S\sigma}{6h}\left(\left[T_{ij}\right]\left\{A^{n+1} f^{1} T^{-1}{}_{01}\right\}_j + \left[T_{ij}\right]\left\{A^{n+1-\rho} f^{1} T^{-1}{}_{\rho 1}\right\}_j\right) \qquad (3)$$

$$J_{i\beta} = -\frac{1}{4S\mu}\left[M_{i\beta}\right] + \frac{S\sigma}{6h}\left[T_{i\beta}\right]\left\{f^{1}\mathrm{T}^{-1}{}_{01}\right\} + \ldots$$

$$-\frac{1}{4S}\frac{\partial}{\partial A_\beta{}^{n+1}}\left(\frac{1}{\mu}\right)\left[M_{ij}\right]\left\{A^{n+1}\right\}_j \qquad (4)$$

where ρ goes from 1 to *back history number* (*BHN*), m goes from 0 to *BHN*, S is the elemental area (m^2), and h stands for the last time increment magnitude (s). M_{ij}, g_{ij}, T_{ij}, f^m and $T_{\rho m}$ are defined as:

$$\left\lfloor M_{ij}\right\rfloor = \left\lfloor g_{2i}g_{2j} + g_{3i}g_{3j}\right\rfloor \qquad (5)$$

$$g_{ij} = \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \qquad (6)$$

$$\left[T_{ij}\right] = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix} \qquad (7)$$

$$f^{m} = \left(-\frac{m}{h^{m}}\right)(0)^{m-1} \tag{8}$$

$$\left[\mathrm{T}_{\rho m}\right] = \left(-\frac{1}{h}\right)\left(\frac{1}{h^{m}}\right)\left(t_{n+1} - t_{n+1-\rho}\right)^{m} \tag{9}$$

The inverse matrix (9) is called in this paper as the BDF damping matrix. Suffixes i, j and n per element are declared as follows $\{i=1, 2, 3\}$, $\{j=1, 2, 3\}$, $\{n=1,2,3\}$. The solution of the nonlinear equation is obtained by applying the Newton-Raphson algorithm (10).

$$J_{in}\Delta A_{n}^{n+1} = -R_{i} \tag{10}$$

A. Initial guess prediction for the A^{n+1} potential

To accelerate the convergence of the Newton-Raphson method in (3) and (4), a prediction for the initial values A^{n+1} is made in accordance with (11).

$$\left\{A_{p}^{n+1}\right\}_{j} = \left\{A_{p}^{n+1-k}\right\}_{j}\eta_{k} \tag{11}$$

where k goes from 0 to BHN and the η constants are calculated as follows,

$$D_{v} = g_{vk}\eta_{k} \tag{12}$$

$$D_{v} = \frac{1}{h^{v}}\left(t_{n+1} - t_{n+1}\right)^{v} \tag{13}$$

$$g_{vk} = \frac{1}{h^{v}}\left(t_{n+1} - t_{n+1-k}\right)^{v} \tag{14}$$

where v goes from 0 to $BHN-1$.

III. SIMULATION RESULTS

The discrete systems given by (3) and (4) were programmed in C language and solved for the domain shown in Fig. 1. The local truncation error considered is given by (15).

$$E = \frac{h}{(t_{n+1} - t_{n-k})}\left[A^{n+1} - A_{p}^{n+1}\right] \tag{15}$$

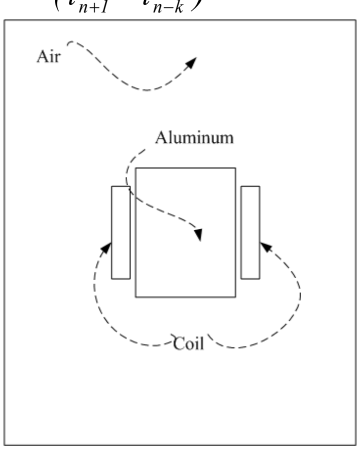

Fig. 1. Domain used in the time-stepping algorithm.

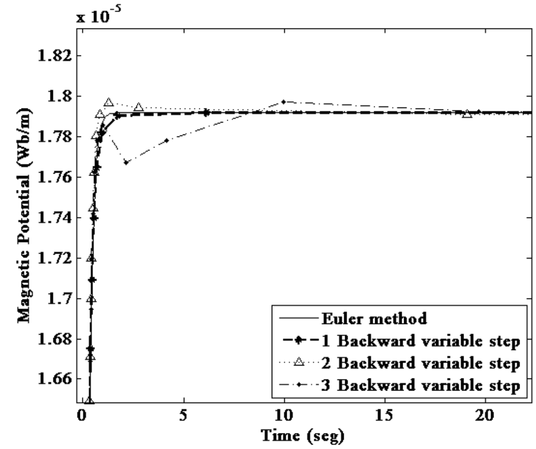

Fig. 2. Transient nodal solution using 1 and 3 backward steps.

The problem was solved employing an absolute error of 1E-5 for one and three backward step history. It was observed that for a low backward step number, the potential results are similar to those obtained using a small time-step size (Euler method). However, the execution time is larger than that obtained with a larger history number. For three and more backward steps, there is a damping problem that will be analyzed in the final version of the paper. A criterion for the selection of the minimum step size will also be included.

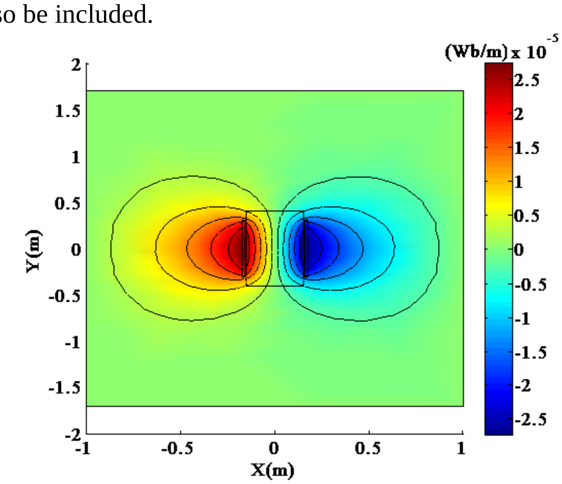

Fig. 3. Magnetic potential distribution at t=20 s, for two backward steps.

IV. REFERENCES

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